

Closed-Back Twists, Counterrotating Twist Tessellations, and Brocard Polygons

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Abstract

Within the world of geometric origami, origami *tessellations* were explored by Fujimoto and Momotani in the 1970s and 1980s, Barreto and Palmer in the 1990s, and numerous others in subsequent years [Gjerde \(2009\)](#). The mathematical underpinnings and mathematical design algorithms for flat-folded origami tessellations have also been of interest; see, for example, [Bateman \(2002\)](#); [Lang and Bateman \(2011\)](#); [Lang \(2011, 2012\)](#); [Demaine et al. \(2016\)](#). The first two of those focused on tessellations constructed via the *shrink-rotate* method, which results in tessellations composed of simple flat twists that all rotate in the same direction—indeed, through the same angle—relative to their underlying tiling. To construct a shrink-rotate tessellation, the underlying tiling must satisfy the *spiderweb condition* [Lang and Bateman \(2011\)](#).

Another form of simple flat twist tessellation is one composed of counterrotating twists, in which twists in adjacent polygons rotate in opposite directions, which includes the famous “Wall” by Momotani [Momotani \(1984, 1997\)](#) and examples by Palmer [Nakayama \(1995\)](#) and Beber [Beber \(2017\)](#), among others.

As with shrink-rotate tessellations, not all tilings can be transformed into a counterrotating twist tessellation. First, and perhaps most obviously, any such tiling must be 2-colorable, i.e., with all vertices of even degree. In fact, the requirements on such tilings are considerably tighter than that, though—and, as we will show, considerably tighter than the spiderweb condition.

Nonetheless, it is possible to identify a large family of tilings that lend themselves to these types of tessellations. In [Lang \(2017\)](#), author RJL analyzed such counterrotating twist tessellations—there called *offset twist tessellations*—and presented several properties and a method of constructing them from an underlying tiling. In this work we present and prove both necessary and sufficient conditions for the construction of a generic counterrotating twist tessellation using the offset twist construction. In addition to the tiling being 2-colorable, sufficient conditions for the construction of a generic counterrotating twist tessellation are that each polygon in the tiling is (a) cyclic (circumscribable by a circle), and (b) Brocard (which we will define).

The Brocard property arises in twists in which the pleats emanating from all sides meet at a single point in the back of the twist, creating a *closed-back twist*, as illustrated in Figure 1. For

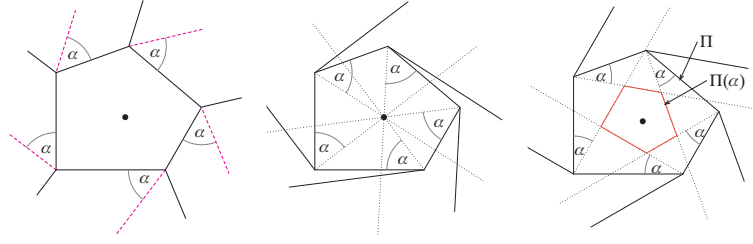


Figure 1: A closed-back twist with twist angle α . Left: crease pattern. Middle: folded form. Dashed lines show hidden edges. Right: folded form for a smaller twist angle α .

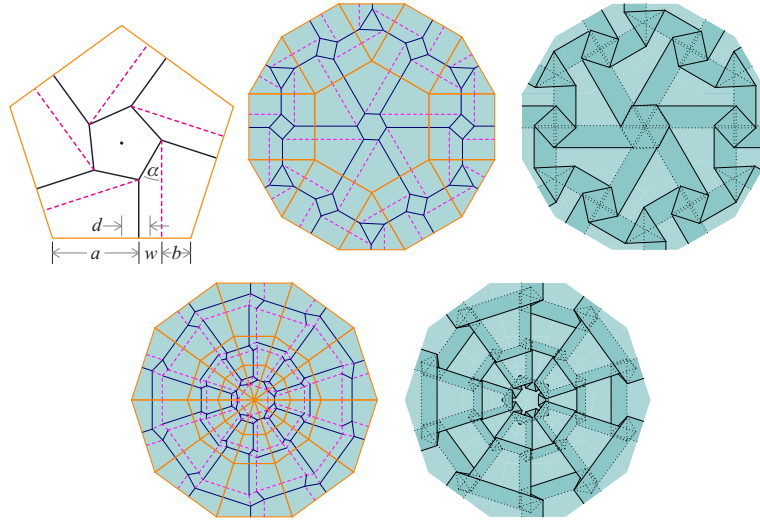


Figure 2: Top left: construction method for an offset twist tile based on parameters (w, d) . Top middle and right: crease pattern and folded form for a tessellation composed of offset twist tiles. Bottom left and right: crease pattern and folded form of a triangle-trapezoid offset twist tessellation.

the twist to fold flat, the angles that the pleats hit the edges of the central polygon must be the same angle all the way around. If the twist is closed-back, then in the folded form, if we draw a line from each vertex making angle α with the adjacent edge, those lines must intersect at a common point, as shown in the center subfigure. If this happens, the point is called the *Brocard point* of the polygon and the polygon is said to be a Brocard polygon.

A method of constructing counterrotating twist tessellations from an initial tiling is shown in Figure 2. One chooses a parameter set (w, d) , then constructs a twist in each tile polygon. The tiles can then be assembled into larger flat-foldable counterrotating twist tessellations. In this paper, we show that for every (w, d) that gives a non-self-intersecting crease pattern, this construction can be applied to any convex polygon, regular or irregular, to give a semifoldable crease pattern (i.e., satisfying Kawasaki's Theorem), if and only if the polygon is both cyclic and Brocard—whether or not the resulting twists are open- or closed-back. That means that such tessellations may be constructed from arbitrary triangles, certain quadrilaterals (squares, certain trapezoids), and higher-order polygons. As an example, we show here a counterrotating twist tessellation with decagonal symmetry, composed of triangle and trapezoidal tiles, all of which are cyclic Brocard, in Figure 2.